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Particle Filtering for Automotive: A survey

Karl Berntorp and Stefano Di Cairano

***Abstract*—Since its introduction more than two decades ago, the particle filter has become an established technique for nonlinear state estimation, due to its capability to cope with severe nonlinearities and non-Gaussian noise. More recently, there has also been rapid development in particle filtering for learning, either for real-time estimation of unknown parameters in the system, or for offline system identification. Due to the increased flexibility the particle filter has been applied in such diverse areas as meteorology, medical imaging, video analysis, robotics, self-driving cars, and aerial vehicles. In this paper, we survey particle filters in vehicle applications, with particular focus on autonomous vehicles. We describe the particle filter and its relation to marginalization and conjugate priors, which are key enablers in several important applications. Based on our own research, we present three recent use cases of particle filtering in the automotive industry and give an outlook on potential research directions.**

I. Introduction

The increased demand for autonomous systems in various forms has scaled up the needs for sensing and estimation techniques that can support the increasingly complex control systems that they are to be integrated with [1], [2]. Self-driving cars in the highest level of autonomy, as an example, should be able to provide full-time operation of all aspects of driving under different roadway and environmental conditions. However, enabling more autonomous features implies that more information needs to be extracted from the sensor data, which requires capable estimation algorithms. The ability to share sensor information between control systems (e.g., through the CAN bus [3]), together with vehicles being equipped with sophisticated sensors such as lidars and cameras, has opened possibilities to provide the information necessary for the control stack to take sensible actions [1], [4], [5]. However, the sensor data are noisy and prone to errors stemming from various sources. Altogether, this puts high requirements on the estimation algorithms to produce estimates that are both robust to the increasingly complex estimation models and the error sources causing imperfect sensor measurements. At the same time as autonomy highly increases the demands on robustness and high performance of the estimation algorithms, due to regulations, cost considerations, and other factors, current production automotive control systems are real-time systems with highly limited computing capabilities of the micro-controllers that execute the estimation and control algorithms [6]. Hence, the estimation algorithms must have high performance and limited computational requirements.

Starting with the work in [7], the theory of particle filters, or more generally sequential Monte-Carlo methods, has evolved over the last two decades such that particle filters nowadays constitute a powerful set of tools for inference in highly nonlinear and possibly non-Gaussian systems [8], [9], such as those encountered in autonomous systems. Particle filters are sampling-based estimators that approximate the posterior density function of the variables of interest conditioned on the measurement history [7], [9]–[11], by representing the posterior density function as a weighted particle system. In the beginning, particle filters were mostly applied to state estimation, but have recently been proven to be powerful methods for learning (system identification), both for real-time parameter estimation [12] and as a key component in particle Markov chain Monte-Carlo methods [13]–[15]. Particle filters have been successfully applied to numerous relevant problems; for example, simultaneous localization and mapping (SLAM) in robotics [16]–[19]; positioning of sea and airborne vessels [11], [20], [21]; and object tracking [22]–[24]. For an account of some of the early applications, see [8]. Particle filters are considered relatively computationally heavy. However, it is interesting to note that according to [11], real-time capabilities of the particle filter were shown in a real-world vehicle positioning application already in 2001, where a particle filter using 15000 particles executed on a small handheld computer. Due to its ability to efficiently cope with complex and inherently uncertain, possibly multimodal, systems, particle filters have been an integral part in several successful automotive applications. Particle filters were important parts of the vehicles competing in the Darpa Grand and Urban challenges [25], [26], and were used for camera and laser based road following [27], object tracking [28], [29], and map-aided localization [30]. The use of particle filters in automotive has since then increased and recent use cases include threat assessment [31], traffic estimation [32], and motion planning [33].

This paper surveys the role of particle filters in automotive applications and gives more insight into some of the most prominent applications. We will go through the main steps in the particle filter and touch upon marginalization [21] and conjugacy [34], [35] in relation to particle filtering, which are key ingredients for enabling efficient implementation. Sec. II discusses particle filtering and some of the key tools for enabling efficient and robust implementations of particle filters. In Sec. III, we survey various automotive applications where particle filters have been used and discuss in slight detail several previous use-cases of particle filtering, and in Sec. IV we give three examples from our own research. Finally, Sec. V gives conclusions and discusses some future research trends as possible enablers of reliable autonomous systems.

II. Particle Filtering

Particle filters are sampling-based estimators consisting of a weighted particle system, where state trajectories are generated

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and weighted according to their consistence with the measurements. Particle filters usually assume discrete-time state-space models relating a hidden state $\boldsymbol{x}_k$ at time instant k to the observation $\boldsymbol{y}_k$,

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_{\boldsymbol{\theta}}(\boldsymbol{x}_k, \boldsymbol{w}_k), \quad \boldsymbol{w}_k \sim p_{\boldsymbol{w}_k}, \tag{1a}$$

$$\boldsymbol{y}_k = \boldsymbol{h}_{\boldsymbol{\theta}}(\boldsymbol{x}_k, \boldsymbol{e}_k), \quad \boldsymbol{e}_k \sim p_{\boldsymbol{e}_k}, \tag{1b}$$

where $\boldsymbol{f}_{\boldsymbol{\theta}}(\cdot)$ is the state-transition function parametrized by $\boldsymbol{\theta}$, $\boldsymbol{h}(\cdot)$ is the observation function, and '$\sim$' means 'distributed according to'. The process noise $\boldsymbol{w}_k$ is stochastic and specified by its density $p_{\boldsymbol{w}_k}$, which typically is known, at least up to some parameters. Similarly, the measurement noise $\boldsymbol{e}_k$ is distributed according to $p_{\boldsymbol{e}_k}$. The distribution of the initial state, $\boldsymbol{x}_0 \sim p_{\boldsymbol{x}_0}$, is also considered known but it is usually not a restriction. A known control input $\boldsymbol{u}_k$ can be straightforwardly included into (1). In terms of density functions, the state-space model (1) can be expressed as

$$\boldsymbol{x}_{k+1} \sim p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k), \tag{2a}$$

$$\boldsymbol{y}_k \sim p(\boldsymbol{y}_k|\boldsymbol{x}_k). \tag{2b}$$

The objective in particle filtering is to estimate the posterior density function of the variable $\boldsymbol{z}_k$ conditioned on the measurement history $\boldsymbol{y}_{0:k} = \{\boldsymbol{y}_0, \cdots, \boldsymbol{y}_k\}$, where $\boldsymbol{z} = \boldsymbol{x}$ for (state) inference and $\boldsymbol{z} = \{\boldsymbol{x}, \boldsymbol{\theta}\}$ for simultaneous inference and learning. Introducing $\boldsymbol{z}$ provides the possibility to estimate parts of either the motion model (1a) or measurement model (1b), or both. For simplicity, here we will use $\boldsymbol{z} = \boldsymbol{x}$ but the particle filter handles both cases. Oftentimes $\boldsymbol{w}_k$ and $\boldsymbol{e}_k$ are assumed independent such that $p(\boldsymbol{w}_k, \boldsymbol{e}_m) = p(\boldsymbol{w}_k)p(\boldsymbol{e}_j)$. From this assumption, the Markov property of (1) implies that $p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_{0:k}, \boldsymbol{y}_{0:k}) = p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)$ and $p(\boldsymbol{y}_k|\boldsymbol{x}_{0:k}, \boldsymbol{y}_{0:k-1}) = p(\boldsymbol{y}_k|\boldsymbol{x}_k)$, which are simplifications that are leveraged frequently in derivation of particle filter algorithms.

Particle filters target the posterior distribution $p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k})$. The recursive update equation for this density is [36], [37]

$$p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k}) = \frac{p(\boldsymbol{y}_k|\boldsymbol{x}_k)}{p(\boldsymbol{y}_k|\boldsymbol{y}_{0:k-1})} p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k-1}), \tag{3}$$

where

$$p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k-1}) = p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) p(\boldsymbol{x}_{0:k-1}|\boldsymbol{y}_{0:k-1}). \tag{4}$$

Particle filters use the weighted particle approximation

$$p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k}) \approx \hat{p}(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k}) = \sum_{i=1}^{N} q_k^i \delta(\boldsymbol{x}_{0:k} - \boldsymbol{x}_{0:k}^i), \tag{5}$$

where q_k^i is the weight of the ith trajectory $\boldsymbol{x}_{0:k}^i$, $\delta(\cdot)$ is the Dirac delta mass, and $\sum_{i=1}^{N} q_k^i = 1$. Particles are propagated forward by sampling the next state. However, instead of sampling directly from (4), which is difficult, particle filters use the concept of importance sampling [38] and introduce a user-designed proposal density to sample from $\boldsymbol{x}_{k+1}^i \sim \pi(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k, \boldsymbol{y}_{k+1})$. By using the proposal density in (3)–(5) and identifying terms,

$$q_k^i = \frac{1}{c_k} \frac{p(\boldsymbol{y}_k|\boldsymbol{x}_k^i) p(\boldsymbol{x}_k^i|\boldsymbol{x}_{k-1}^i)}{\pi(\boldsymbol{x}_k^i|\boldsymbol{x}_{k-1}^i, \boldsymbol{y}_k)}, \tag{6}$$

where c_k is a normalization constant. There are many different implementation aspects that are crucial to consider in any realistic implementation, such as resampling, numerical evaluation of the weights, divergence monitoring, and dependent noise. For coverage of those topics, see for example [37], [39]–[41]. A key aspect of particle filters is the choice of proposal $\pi(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}, \boldsymbol{y}_k)$, because the proposal determines how well the predicted particles will reflect the distribution to be estimated. A widely used proposal is the prior, that is, $\pi(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}, \boldsymbol{y}_k) = p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})$. This choice ignores the information provided by the current measurement and is clearly suboptimal. However, it is rather intuitive and, moreover, it leads to a particularly simple weight update (6),

$$q_k^i = \frac{1}{c_k} p(\boldsymbol{y}_k|\boldsymbol{x}_k^i). \tag{7}$$

There is a wide range of proposals that have been exploited in literature, see [10], [37] for a few of them. Algorithm 1 gives a basic implementation of the particle filter. The particle filter has complexity $\mathcal{O}(N)$, N being the number of particles, but a concern is the poor scaling with the state dimension, which prohibits the use of the standard formulation of particle filters in many realistic automotive applications. Fortunately, there are several techniques available to allow particle filtering in higher-dimensional spaces, two of which are introduced next.

Algorithm 1 Particle filter algorithm

Generate $\{\boldsymbol{x}_0^i\}_{i=1}^N \sim p_{\boldsymbol{x}_0}$ and set $\{q_0^i\}_{i=1}^N = 1/N$.
for $k \leftarrow 0$ **to** T **do**
  Determine weights using (6) for $i \in \{1, \ldots, N\}$.
  Optionally, draw with probability q_k^i N samples from $\{\boldsymbol{x}_k^i\}_{i=1}^N$ and set $\{q_k^i\}_{i=1}^N = 1/N$.
  Generate $\boldsymbol{x}_{k+1}^i \sim \pi(\boldsymbol{x}_{k+1}^i|\boldsymbol{x}_k, \boldsymbol{y}_{k+1})$, $i \in \{1, \ldots, N\}$.
end for

A. Marginalization

To reduce computational complexity, it is advantageous to exploit model structure. This is the idea behind marginalization, where the subset of the state space that allows for analytic expressions is marginalized out [21], [42]. Marginalized particle filters rely on the decomposition of a vector of variables into $\boldsymbol{z}_k = [\boldsymbol{x}_k \ \boldsymbol{\eta}_k]^{\mathrm{T}}$ and the corresponding decomposition of the posterior density as

$$p(\boldsymbol{z}_{0:k}|\boldsymbol{y}_{0:k}) = p(\boldsymbol{\eta}_k|\boldsymbol{x}_{0:k}, \boldsymbol{y}_{0:k}) p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k}), \tag{8}$$

where $p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k})$ is approximated with a particle filter, which involves marginalizing (integrating out) $\boldsymbol{\eta}_k$ from the particle filter. The density $p(\boldsymbol{\eta}_k|\boldsymbol{x}_{0:k}, \boldsymbol{y}_{0:k})$ is computed analytically by exploiting structure, either model structure or assumptions on the prior distribution of $\boldsymbol{\eta}_k$. A common and fairly general model structure that appears in many automotive estimation problems is

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{A}(\boldsymbol{x}_k)\boldsymbol{\eta}_k + \boldsymbol{F}(\boldsymbol{x}_k)\boldsymbol{w}_k^{\boldsymbol{\eta}}, \tag{9a}$$

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{\eta}_k + \boldsymbol{G}(\boldsymbol{x}_k)\boldsymbol{w}_k^{\boldsymbol{x}}, \tag{9b}$$

$$\boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{C}(\boldsymbol{x}_k)\boldsymbol{\eta}_k + \boldsymbol{e}_k, \tag{9c}$$

where all involved noise sources are assumed Gaussian distributed, although for some special cases this assumption may be excessively restrictive [21]. For a given $\boldsymbol{x}_k$, (9) is a linear Gaussian model. Hence, (9) makes it possible to estimate the distribution of $\boldsymbol{\eta}$ with N Kalman filters, where each Kalman filter is conditioned on a state trajectory $\boldsymbol{x}^i_{0:k}$. The more states that can be analytically estimated the better [21], [40], and a special case of the marginalized particle filter can be found in state-of-the-art SLAM methods [16]–[18], where the state vector contains many thousand variables. Marginalized particle filters using (9) have frequently been used in the automotive applications, including slip estimation [43], road and target tracking [40], [44]–[46], and vehicle positioning [37].

B. Conjugate Priors in Particle Methods

Together with marginalization, conjugate priors [34], [47] are key for some recent applications of joint state and parameter estimation in the automotive domain [48]–[50]. A typical use-case for conjugate priors in particle filtering is to utilize the decomposition (8), with $\eta = \boldsymbol{\theta}$,

$$p(\boldsymbol{z}_{0:k}|\boldsymbol{y}_{0:k}) = p(\boldsymbol{\theta}_k|\boldsymbol{x}_{0:k}, \boldsymbol{y}_{0:k})p(\boldsymbol{x}_{0:k}|\boldsymbol{y}_{0:k}). \tag{10}$$

If a prior distribution belongs to the same family as the posterior distribution, the prior is conjugate to the likelihood. Conjugate priors are useful because they allow closed-form computations of the posterior densities [12], [34], [35], which, similarly to the marginalized particle filter, increases estimation performance and at the same time reduces computational load. There are different prior distributions that are conjugate for different likelihoods, but perhaps the most common prior distribution is the inverse-Wishart prior. For multivariate Gaussian distributed data $\bar{\boldsymbol{w}} \in \mathbb{R}^d$ with unknown mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$, a Normal-inverse-Wishart distribution defines the conjugate prior [34], $p(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) := \mathcal{NIW}(\gamma_k, \hat{\boldsymbol{\mu}}_k, \boldsymbol{\Lambda}_k, \nu_k)$, through the hierarchical model

$$\begin{aligned}\boldsymbol{\mu}_k|\boldsymbol{\Sigma}_k &\sim \mathcal{N}(\hat{\boldsymbol{\mu}}_k, \boldsymbol{\Sigma}_k),\\ \boldsymbol{\Sigma}_k &\sim \mathcal{IW}(\nu_k, \boldsymbol{\Lambda}_k)\\ &\propto |\boldsymbol{\Sigma}_k|^{-\frac{1}{2}(\nu_k+d+1)} e^{\left(-\frac{1}{2}\mathrm{tr}(\boldsymbol{\Lambda}_k\boldsymbol{\Sigma}_k^{-1}\right)},\end{aligned}$$

where $\mathrm{tr}(\cdot)$ is the trace operator. Thus, for a Gaussian distributed likelihood and a $\mathcal{NIW}$ distributed prior, the posterior is also $\mathcal{NIW}$ distributed. An example of another frequently used conjugate prior is the gamma distribution, which is a conjugate prior for an unknown Poisson rate and has been used in extended object tracking for cars using lidar [23].

III. Particle Filtering in Automotive Applications

In this section we survey some important applications of particle filtering in automotive applications. We highlight different types of applications, ranging from high level estimation of groups of vehicles to estimation of specific parameters in the vehicle dynamics models.

A. Positioning

Particle filters were adopted early on for automotive applications. They were used quite substantially in the Darpa Grand and Urban challenges [25]–[30]. However, the history of particle filters in automotive started earlier than that [51]. An early driver for adoption of particle filters were positioning applications. Contributions in this area are [37], [51], which consider map-aided positioning. In [51], the main difficulty of the estimation problem is not the motion model, since it is linear and consists of the position integrated using the velocity, which is measured (e.g., from the wheel speeds). Instead, the measurement function is highly nonlinear and is composed of a road map and GPS or base station measurements. Subsequently, map matching [52] is used to map a position measurement onto the road. This is a highly multimodal estimation problem, especially in the transient phase. Using marginalization [37], it is possible to include more sophisticated motion models while retaining nice computational properties, where some states are analytically computed and the few remaining ones are treated in the particle filter. This reasoning has been extended in [53], where a general framework for map-aided positioning for both indoor and outdoor areas is presented. The main advantage of using a particle filter is that the entire multimodal probability density is exploited to get accurate positioning. Furthermore, modern camera based vehicles can be equipped with traffic sign recognition, which can be used together with a sign location database as an additional measurement.

B. Traffic State Estimation

Another application where particle filters have been largely researched is in traffic state estimation (TSE), which refers to inference of variables associated with traffic, such as flow, density, and speed, see [32], [54] for two overview papers. Due to the increased automation and more sophisticated traffic operations for large traffic networks and the need to limit effect of congestions, the importance of TSE is a heavily researched field. TSE is a complex problem that lends itself very well to particle filtering because of its ability to handle nonlinear multimodal systems, and it is quite common to employ macroscopic models, which are models that represent the average traffic behavior [55]. A problem is that the number of variables to estimate is considerably larger than the number of variables that are measured, since the sensors are usually located at the boundaries of traffic segments. Typical sensor measurements include radar detectors and cameras. For an overview of different models, see [32]. In [56], a mixture Kalman filter is employed using models of the traffic density. This has been extended in [57] to also consider speed and uses a particle filter to estimate the density and speed of the flow. Particle filters lend themselves well to parallellization, and this has been explored in [58]. Work along increasing performance in very sparse sensor environments is found in [59]. A test study on data from Chicago's interstate I-55 highway is done in [60]. Here, a particle filter methodology is applied to estimate the state of the density, as well as incorporating a MCMC sampler for estimating the parameters involved in the employed models.

C. Object Tracking

Object tracking is a rich field that dates back to the second world war with extensive research that includes several types of tracking problems,

- Point object tracking: Each object generates at most a single measurement per time step.
- Extended object tracking: Each object generates multiple measurements per time steps and the measurements are spatially structured around the objects.
- Group object tracking: Each object generates multiple measurements per time step, and the measurements are spatially structured around the object. A group object consists of two or more subobjects that share some common motion, which implies that the objects are tracked in groups.

Extended object tracking is an increasingly common tracking problem in automotive due to the increased resolution of onboard sensors such as radar and lidar, which generate multiple measurements of an object. For an overview of extended object tracking in general and particle filters in particular, see [23] and [61], respectively. The minimum set of states to estimate in exended object tracking is the position and heading of the vehicle, in addition to its geometry (e.g., length and width for rectangular objects). Due to the multiple hypotheses and multiple possible targets available, this problem is usually multimodal. At the Urban Grand challenge, Stanford's vehicle Junior was equipped with a marginalized particle filter based object tracking module that used 3D range data to execute a particle filter at 40 Hz [45], with successful execution for a number of nontrivial situations. More recent applications are the work from Denso [62], which considered multi-target tracking using automotive fast chirp modulation (FCM) radars using particle filtering, the Guassian process convolution particle filter with nonregular shape [63], where the shape of the object is modeled using a Gaussian process, and the evaluation study [64], which compared different approaches on radar and lidar data for autonomous driving applications. Gaussian mixture approximations of the measurement likelihood, which are reminiscent of traditional particle filters, are used in numerous target tracking applications [65], [66].

D. Parameter Estimation

Recently, efficient particle-filter based techniques for estimating various parameters associated with vehicle safety have started to emerge. There are well established vehicle models for many different types of vehicle control applications [67]–[70], but oftentimes several of the parameters included in the models are uncertain at runtime, which makes sophisticated control a hard task. Parameter estimation can be done by augmenting the vehicle state with the parameters of interest. However, this implies introducing artificial (incorrect) dynamics of the parameter evolution, and it also implies increased computational load, since the state vector grows significantly in size. With the recent developments in adaptive particle filtering and particle Markov-chain Monte-Carlo methods that leverage marginalization and conjugate priors, efficient and cheap software solutions can be developed. An interesting application of adaptive particle filters is tire pressure monitoring [71]. It is possible to equip the wheels with pressure sensors, however, this is expensive and error prone. To this end, [48] proposed an approach where the tire radii are treated as external disturbances acting on a nonlinear kinematic vehicle model. Using conjugate priors, the radii can be integrated out from the state equations, resulting in an efficient implementation where the particle filter estimates the position and heading of the vehicle using GPS measurements. In Sec. IV we will show two examples from our own research leveraging adaptive particle filtering for real-time tire-friction estimation [50], [72] and sensor calibration [49], [73]. Based on recent PMCMC methods [13], [14], we developed a method for offline tire calibration with convergence guarantees [74].

E. Threat Assessment

Threat assessment and the accompanying theme of motion prediction is the task of predicting future behavior of traffic participants and assessing the risk of, for instance, impact. In a probabilistic context, the considered problem is highly nonlinear and non-Gaussian and can be confronted in several ways, where sequential Monte-Carlo is one possible approach [31]. Work in this direction is [75], where Monte-Carlo sampling was used for determining the probability of collision. A related approach is found in [76], where reachable sets of vehicles were computed using a biased driver-preference distribution incorporated into a Monte-Carlo framework. A comparison study including Monte-Carlo approaches were undertaken in [77], and particle filtering with driver-preference proposal was used in [78] for predicting the behavior of traffic participants.

IV. Recent Automotive Applications

In this section we highlight three successful applications of particle-filter techniques based on our own research.

A. Road-Friction Estimation

Reliable knowledge of at least parts of the tire-road friction function is critical in several control systems for autonomous driving capabilities [79]–[82]. The tire stiffness is the initial linear slope of the tire-force curve, when expressing the tire force as a function of wheel slip (either longitudinal or lateral). Estimation of the whole force-slip curve is intractable in online implementations. However, knowledge of the slope of the force curve provides information on the available total friction. For production vehicles, real-time estimation of the tire stiffness is challenging because the available sensors only measure a subset of the vehicle state, they are error prone and noisy, and the vehicle dynamics, whose state are also only partially measured, is largely affected by the tire stiffness, which is highly uncertain and varies with surface. Here, we opt for a sensor-fusion approach and estimate the vehicle state jointly with the tire stiffness components of front and rear tire, assuming a single-track vehicle model. The states to estimate are longitudinal and lateral velocity, and yaw rate, that is, $\boldsymbol{x} = [v^X \;\; v^Y \;\; \dot{\psi}]^{\mathrm{T}}$. However, we assume normal driving conditions, meaning that the tire forces can be expressed as

$$F^x \approx C^x \lambda, \quad F^y \approx C^y \alpha, \tag{11}$$

where C^x and C^y are the longitudinal and lateral stiffness, respectively. Inserting (11) into the equations of motion for the single-track model gives

$$m(\dot{v}^X - v^Y\dot{\psi}) = C_f^x\lambda_f + C_r^x\lambda_r - C_f^y\alpha_f\delta, \tag{12a}$$

$$m(\dot{v}^Y + v^X\dot{\psi}) = C_f^y\alpha_f + C_r^y\alpha_r + C_f^x\lambda_f\delta, \tag{12b}$$

$$I\ddot{\psi} = l_fC_f^y\alpha_f - l_rC_r^y\alpha_r + l_fC_f^x\lambda_f\delta. \tag{12c}$$

The wheel slip and the slip angles are computed as

$$\lambda := \frac{v^X - R_w\omega}{v^X} \quad \text{if } v^X \geq R_w\omega,$$

$$\lambda := \frac{v^X - R_w\omega}{R_w\omega} \quad \text{if } R_w\omega > v^X,$$

$$\alpha_f \approx \delta - \frac{v^Y + l_f\dot{\psi}}{v^X}, \quad \alpha_r \approx \frac{l_r\dot{\psi} - v^Y}{v^X},$$

where ω is the wheel rotation rate and R_w is the effective wheel radius. The wheel rotation rates and steering angle are treated as known inputs. This is consistent with many navigation systems, where dead reckoning is used to decrease state dimension. We treat the tire stiffness parameters as deviations from a nominal component,

$$C^x \approx C_n^x + \Delta C^x, \quad C^y \approx C_n^y + \Delta C^y, \tag{13}$$

where C_n is the nominal value of the stiffness. We consider a front-wheel drive vehicle and the process noise $\boldsymbol{w}_k = [\Delta C_f^x \ \Delta C_f^y \ \Delta C_r^y]^\mathrm{T}$ Gaussian distributed according to $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$, with $\boldsymbol{\mu}_k$ and $\boldsymbol{\Sigma}_k$ unknown. Hence, by putting a normal-inverse-Wishart prior on the mean and variance of the process noise, we can utilize conjugate priors and estimate the tire stiffness using adaptive particle filtering techniques [50]. We use the longitudinal and lateral accelerations a_m^X, a_m^Y, and the yaw rate $\dot{\psi}_m$ as measurements, forming the measurement vector $\boldsymbol{y}_k = [a_m^X \ a_m^Y \ \dot{\psi}_m]^\mathrm{T}$. The measurement model has the same form as (9c). Altogether, the estimation problem consists of three vehicle states, three mean values of the Gaussian process noise, and a symmetric 3×3 matrix. Estimating all these states in a naive particle filter implementation would be computationally prohibitive. However, since we leverage conjugate priors and marginalize out the parameters from the particle filter equations, the particle filter only estimates the three vehicle states, and the rest is done using closed-form expressions [50].

We have used a mid-size SUV, equipped with industry-grade validation equipment to gather data, and collected several different data sets, on both snow and asphalt. The parameters of the vehicle model and the tire-stiffness parameters are extracted from data sheets and extensive experimental validation.

The data set consists of normal driving on a regular dry asphalt road and is about 400 s long. This data set is collected from a period of regular driving on a standard two-lane road and the test was not specifically designed for this experiment. The road requires only light steering, which reduces observability, and it has nonzero inclination and bank angles, which is not explicitly accounted for in the current implementation. Thus, the dataset also tests how robust the algorithm is to these unmodeled effects.

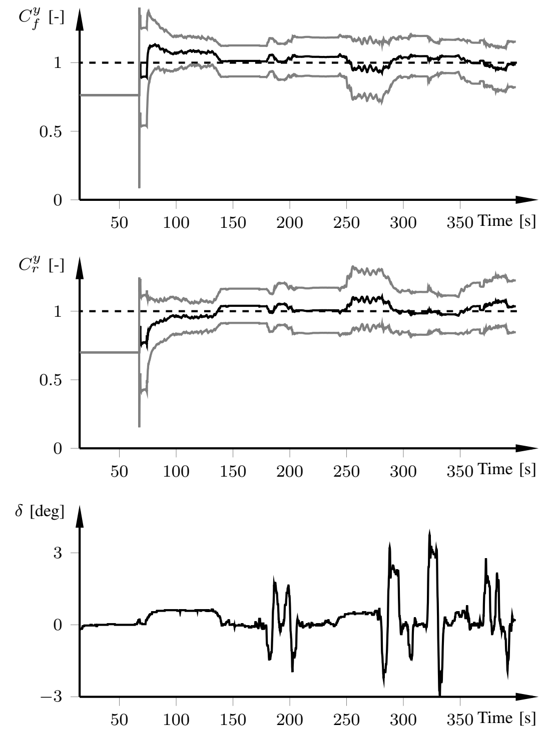

Fig. 1. Upper two plots show the estimated normalized tire stiffness (black) and associated standard deviation (gray) using 500 particles on one realization for the data set collected on dry asphalt. The true values are in dashed. Lowest plot shows the steer angle δ.

For the results, we use $N = 500$ particles. On a dSPACE MicroAutoBox-II rapid prototyping unit, this results in a computation time of slightly less than 5 ms, which scales linearly with the number of particles. Fig. 1 displays the results for one realization. The first 67 s of the experiment consists of constant-speed driving on a straight road, and, because of that, the estimator is inactive. At activation, the uncertainty of the estimate (gray) is initially large but decreases rapidly until it approximately reaches steady state.

B. Sensor Calibration

The recent surge for enabling novel autonomous capabilities [1], [2], [83], [84] induces the need for sensor information that can be used over longer time intervals to reliably predict the vehicle motion. However, production vehicles are typically equipped with low-cost sensors that are prone to time-varying offset and scale errors, and may furthermore have relatively low signal-to-noise ratio [4]. For instance, the lateral acceleration and heading-rate measurements are known to have significant drift and noise in the sensor measurements, leading to measurements that are only reliable for prediction over a very limited time interval. Similarly, the sensor measuring the steering-wheel angle has an offset error that, when used for dead reckoning in a vehicle model, leads to prediction errors that accumulate over time.

By leveraging conjugate priors and marginalization, in this application we estimate the bias and noise variances of the inertial sensors (acceleration and heading rate) and steering wheel sensor, which are often considered the most important sensors for ADAS applications, the steering wheel because it is the main actuator of the vehicle in terms of lateral dynamics,

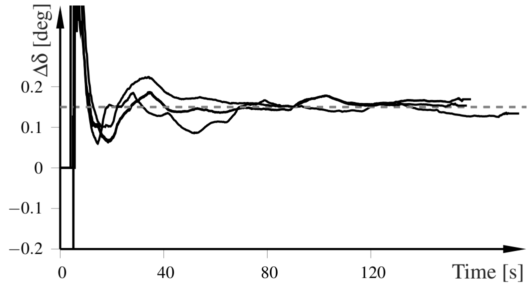

Fig. 2. The estimated steering-wheel offset (black solid) for four different data sets and the ground truth (gray dashed), as obtained by an offline optimization-based procedure, in experiments.

and the inertial sensors because they are related to the vehicle speed and heading through integration, which causes large drift when bias is present.

We use the same vehicle model (12) and measurement setup as in Sec. IV-A. However, in this scenario we model the steering wheel measurement, which is an input to the vehicle model, and the measurement noise as Gaussian distributed with unknown mean and covariance. Again, by leveraging the normal-inverse-Wishart as conjugate prior and marginalization, we can decrease the dimensionality of the particle filter considerable, thus enabling real-time feasibility.

Fig. 2 displays the estimated steering-wheel angle offset for four different data sets. We have used the same experimental setup as in Sec. IV-A. After the initial transients, the estimated bias converges to values very close to the true offset for all data sets. This indicates that the method is reliable for different types of driver behaviors. The true steering offset has been determined from an offline optimization procedure using a PMCMC method, see [74] where it was used for calibration of tire-friction parameters.

C. Motion Planning

The objective of a motion planner is to determine a motion plan and the corresponding input sequence over a planning horizon T_f to navigate the road safely while satisfying input constraints, road constraints, and obstacle constraints, while typically minimizing some objective function [1]. Formulated in a statistical framework, this problem is nonlinear (due to the vehicle kinematics and dynamics) and non-Gaussian (due to obstacles, environment, and multiple solutions), which is exactly the type of problems particle filters are tailored for. Hence, by interpreting the motion-planning problem as an estimation problem, we can apply particle filtering for solving this complex task [33], [85], [86].

For a motion planner operating normal driving maneuvers, we can use a kinematic nonlinear single-track model, which can be written as (1a), and we model the objective of driving, such as safety distances, mid-lane deviation, and preferred vehicle speed, as nonlinear probabilistic driving requirements on the form (1b), where the measurement noise takes the form of allowed deviations from the objectives.

We have conducted extensive validation in full-size vehicle tests and on a scaled vehicle platform [87] to verify the approach. Fig. 3 shows snapshots of one such experiment in the scaled vehicle platform for a 135 s excerpt of an eight minutes long data set, where there are two other vehicles, one in each lane, in front of the ego vehicle initially driving with considerably slower speed than the ego vehicle. The other vehicles change their respective speeds at time instants unknown to the ego vehicle. First, a trajectory that makes the vehicle to slow down to satisfy the predefined safety distance is computed ($t = 121$ s). The motion planner computes trajectories for both lanes, but determines that it is preferable to stay in the inner lane. Between $t = 121$ s and $t = 160$ s, the other vehicle in the outer lane has increased its speed, which makes the motion planner to change lane. At $t = 199$ s the other vehicle in the inner lane speeds up, which leads the motion planner to decide to change lane again. The motion planner determines that it is safe to overtake the other vehicle in the inner lane ($t = 213$ s). Finally, the ego vehicle moves back to the preferred lane. In several of the snapshots, the non-Gaussianity of the estimation problem is clearly seen, where the distribution of the particles spread out in two distinct paths.

V. Conclusion

The advancements in particle filter theory in combination with more sophisticated sensors, improved computing platforms, and increasing demands on the estimation algorithms led to particle filters being used in many different automotive applications. An early application was positioning, but as discussed in the paper, particle filters are now used in diverse areas such as traffic state estimation, object tracking, motion planning, threat assessment, and road surface estimation. The demand for more capabilities on the sensing and estimation side is likely to increase in coming years, and therefore new opportunities for particle filtering will appear. For example, with the increased communication between vehicles and the possibility to perform computations remotely [53], tasks that are not real-time critical but still important for proper operation of the vehicle can preferably be done remotely and then transferred to the vehicle. This can open up more possibilities for batch methods such as some of the recent developed model-based learning methods, where particle filters play an important role [13]–[15], [88].

Particle filters have shown large potential for automotive applications where Gaussian-assumed type filters do not suffice. They are easy to implement and with adaptive particle filtering techniques, they are easy to tune. However, there are still several challenges remaining. First, particle filters are still relatively computationally complex, especially for higher-dimensional spaces unless model structure, such as linear Gaussian substructures [21], can be exploited. Here, enabling software-based services for edge computing and parallellization of particle filters are two promising ideas [58].

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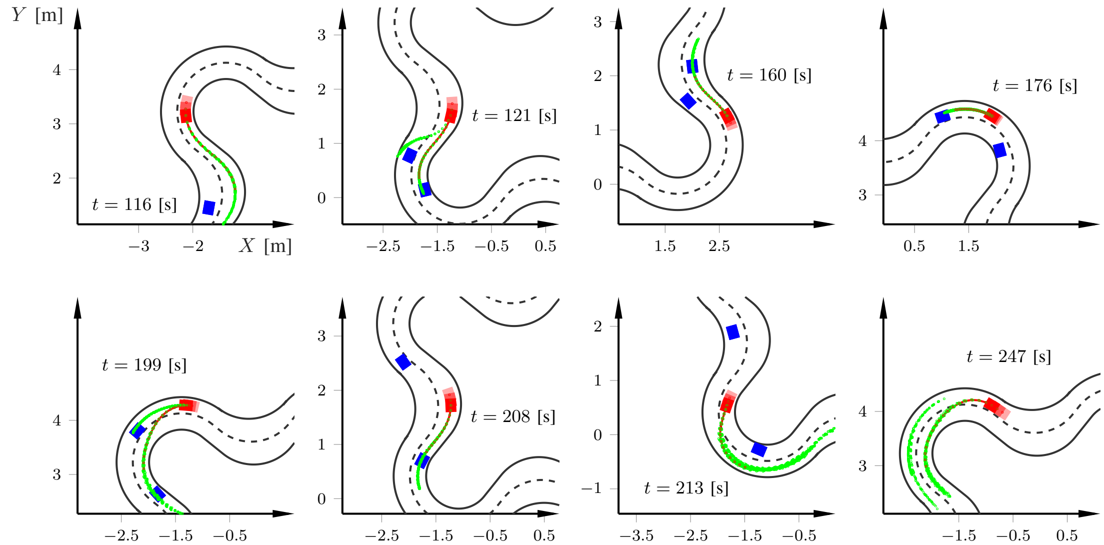

Fig. 3. Eight snapshots from experimental validation particle filter based motion planning. The ego vehicle in red, obstacles in blue, and particles from the particle filter based motion planner in green. In every figure, snapshots are shown every 0.5 s in increasingly darker colors.

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