MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

A Brief Theory of Guided Signal Reconstruction

Knyazev, A.; Gadde, A.; Mansour, H.; Tian, D.

TR2017-088 July 2017

Abstract

An axiomatic approach to signal reconstruction is formulated, involving a sample consistent set, defined as a set of signals sample-consistent with the original signal, and a given guiding set, describing desired reconstructions. New frame-less reconstruction methods are proposed, based on a reconstruction set, defined as a shortest pathway between the sample consistent set and the guiding set, where the guiding set is a closed subspace and the sample consistent set is a closed plane in a Hilbert space. Existence and uniqueness of the reconstruction set are investigated. Connections to earlier known consistent, generalized, and regularized reconstructions are clarified, and new and improved reconstruction error bounds are derived.

International Conference on Sampling Theory and Applications (SampTA)

This work may not be copied or reproduced in whole or in part for any commercial purpose. Permission to copy in whole or in part without payment of fee is granted for nonprofit educational and research purposes provided that all such whole or partial copies include the following: a notice that such copying is by permission of Mitsubishi Electric Research Laboratories, Inc.; an acknowledgment of the authors and individual contributions to the work; and all applicable portions of the copyright notice. Copying, reproduction, or republishing for any other purpose shall require a license with payment of fee to Mitsubishi Electric Research Laboratories, Inc. All rights reserved.

Copyright © Mitsubishi Electric Research Laboratories, Inc., 2017
201 Broadway, Cambridge, Massachusetts 02139

A Brief Theory of Guided Signal Reconstruction*

Andrew Knyazev, Hassan Mansour, Dong Tian
Mitsubishi Electric Research Laboratories
Cambridge, MA 02139
{knyazev, mansour, tian}@merl.com

Akshay Gadde
Department of Electrical Engineering
University of Southern California
Los Angeles, CA 90089
agadde@usc.edu

***Abstract*—An axiomatic approach to signal reconstruction is formulated, involving a sample consistent set, defined as a set of signals sample-consistent with the original signal, and a given guiding set, describing desired reconstructions. New frame-less reconstruction methods are proposed, based on a reconstruction set, defined as a shortest pathway between the sample consistent set and the guiding set, where the guiding set is a closed subspace and the sample consistent set is a closed plane in a Hilbert space. Existence and uniqueness of the reconstruction set are investigated. Connections to earlier known consistent, generalized, and regularized reconstructions are clarified, and new and improved reconstruction error bounds are derived.**

I. Introduction

Signal reconstruction is a standard technique that arises naturally in signal processing and machine learning. A classical example is reconstruction of band-limited signals from their time-domain samples. Recently, the reconstruction of signals on graphs from signal samples on a subset of nodes of the graph has been gaining popularity, e.g., [1], [2], and finds applications in graph based semi-supervised learning; see, e.g., [3]. In this context, the signals are considered to be band-limited with respect to eigenvalues of a graph Laplacian.

A Hilbert space framework allows investigating signal reconstruction in a general and concise manner. To this end, we consider a problem of determining a reconstruction $\hat{\mathbf{f}} \in \mathcal{H}$ of an unknown original signal $\mathbf{f} \in \mathcal{H}$ from a sample of $\mathbf{f}$, where $\mathcal{H}$ is a Hilbert space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and a corresponding norm $\| \cdot \|$. The sampling of $\mathbf{f}$ is defined as an orthogonal projection $\mathbf{Sf}$ on a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ called the *sampling subspace*.

The original signal $\mathbf{f}$ is typically not known, only the sampled original signal $\mathbf{Sf}$ is available as an input to a reconstruction method. Since sampling involves loss of information, we need some a priori assumptions on the original signal $\mathbf{f}$ to be recovered. One such assumption may be that the signal $\mathbf{f}$ belongs to a closed subspace $\mathcal{T} \subseteq \mathcal{H}$ that can be thought of as a *target reconstruction subspace*. Alternatively, the signal $\mathbf{f}$ may not lie strictly in $\mathcal{T}$, but may be well approximated by its projection on the subspace $\mathcal{T} \subseteq \mathcal{H}$. We prefer to call $\mathcal{T}$ a *guiding reconstruction subspace*, since in our technique the reconstructed signal $\hat{\mathbf{f}}$ is not necessarily restricted to $\mathcal{T}$. Another example of the prior structure is that the signal $\mathbf{f}$ belongs to a compact subset of $\mathcal{H}$, determined by the "smoothness" of $\mathbf{f}$. In any case, the reconstruction that minimizes the reconstruction error $\|\hat{\mathbf{f}} - \mathbf{f}\|$ is naturally desired.

The guiding set can be determined using a model or other form of description of desirable reconstructed signal behavior, e.g., learned from training datasets. For signals with natural spectral properties, spectral transforms, e.g., Fourier, cosine, and wavelet transforms, can be used to transform signals into a spectral domain, where the guiding subspace can be chosen as corresponding to certain frequency ranges, e.g., assuming that the desired signal is band-limited.

For signals without self-evident spectral properties, the signals are embedded into a specially constructed structure, depending on a type of the signal, e.g., a graph, or a Riemannian manifold, wherein spectral properties are determined by an "energy" norm and its defining operator, e.g., graph Laplacian, or the Laplace-Beltrami operator, correspondingly [5]. The guiding subspace can then be chosen to approximate an invariant subspace of the energetic operator, corresponding to certain ranges in its spectrum, e.g., assuming that the desired signal is band-limited, having components primarily from the low part of the spectrum of the energetic operator. The embedding also involves choosing a distance in the embedded space, depending on a signal similarity measure in the signal space, which can comprise, e.g., correlation, coherence, divergence, or metric, depending on the type of the signal.

A. Notation

Let $\mathbf{S}$ and $\mathbf{T}$ be the orthoprojectors onto the closed subspaces $\mathcal{S}$ and $\mathcal{T}$, respectively. Let $\mathbf{S}^\perp = \mathbf{I} - \mathbf{S}$ and $\mathbf{T}^\perp = \mathbf{I} - \mathbf{T}$, where $\mathbf{I}$ is the identity operator, denote the orthoprojectors onto their orthogonal complements $\mathcal{S}^\perp$ and $\mathcal{T}^\perp$. Let $R(\mathbf{A})$ denote the range of operator $\mathbf{A}$ and $N(\mathbf{A})$ its null space, then, e.g., $\mathcal{S} = R(\mathbf{S})$ and $\mathcal{S}^\perp = R(\mathbf{S}^\perp) = N(\mathbf{S})$.

We sample an element $\mathbf{f} \in \mathcal{H}$ by its projection on $\mathcal{S}$, i.e. the observed sample is given by $\mathbf{Sf}$, and want to reconstruct $\mathbf{f}$ from $\mathbf{Sf}$. The signal $\mathbf{f}$ to be reconstructed can be split into two orthogonal components:

$$\mathbf{f} = \mathbf{Sf} + \mathbf{x}, \text{ where } \mathbf{Sf} \in \mathcal{S} \text{ and } \mathbf{x} \in \mathcal{S}^\perp, \tag{1}$$

where $\mathbf{Sf}$ is the observed sample of $\mathbf{f}$ and $\mathbf{x}$ contains the missing information to be reconstructed.

B. Prior work

Two main kinds of sample consistent reconstructions are known: subspace-based constrained reconstructions using oblique projectors leading to $\hat{\mathbf{f}} \in \mathcal{T}$, e.g., [6]–[8], and energy

*An extended version of this paper is posted as [4].

minimization-based reconstructions, e.g., in [8] and generalized abstract splines [9, Section 4]. Practical reconstruction is usually performed using frames for $\mathcal{S}$ and $\mathcal{T}$, correspondingly. In this context, $\mathcal{S}$ is separable and comes, e.g., with an orthonormal countable frame F. Consequently, $\mathbf{T}F$ is also a frame for $\mathcal{T}$, having the frame operator $\mathbf{TST}$ restricted to $\mathcal{T}$, assuming $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$ and strict positivity of the *minimal gap* [10, Section IV-4] between $\mathcal{S}^\perp$ and $\mathcal{T}$, which makes the inverse of the frame operator bounded. The approach we present in this paper is frame-less, dealing directly with the orthogonal projectors $\mathbf{S}$ and $\mathbf{T}$ onto the subspaces $\mathcal{S}$ and $\mathcal{T}$.

A set of all signals, having the same sample $\mathbf{Sf}$, is a closed plane $\mathbf{Sf}+\mathcal{S}^\perp$ that we call a *consistent plane*. But $\mathbf{Sf}+\mathcal{S}^\perp$ and $\mathcal{T}$ generally do not intersect, in which case no reconstruction $\hat{\mathbf{f}}$ can be constrained to both sets as required in [6], [7].

For a solution, which is in both $\mathcal{T}$ and $\mathbf{Sf}+\mathcal{S}^\perp$, to exist for any $\mathbf{f}$, we need $\mathcal{S}^\perp + \mathcal{T} = \mathcal{H}$. Additionally, for such a solution to be unique we need $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$. Otherwise there can be multiple signals in $\mathcal{T}$ having the same samples. If both of these conditions are satisfied, then a unique sample consistent solution in $\mathcal{T}$ is given by $\mathbf{P}_{\mathcal{T}\perp\mathcal{S}}\mathbf{f}$ where $\mathbf{P}_{\mathcal{T}\perp\mathcal{S}}$ an oblique projector on $\mathcal{T}$ along $\mathcal{S}^\perp$. Non-uniqueness caused by $\mathcal{S}^\perp \cap \mathcal{T} \neq \{\mathbf{0}\}$ can be mathematically resolved by replacing $\mathcal{H}$ with a quotient space $\mathcal{H}/\{\mathcal{T} \cap \mathcal{S}^\perp\}$. After such a replacement, we have $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$, which we assume for the rest of this section.

The assumption $\mathcal{S}^\perp + \mathcal{T} = \mathcal{H}$ can be disadvantageous and very restrictive in applications. Even if $\mathbf{Sf} + \mathcal{S}^\perp$ and $\mathcal{T}$ do intersect, finding their intersection numerically may be difficult,

The generalized reconstruction methods from [11] and the minimax regret in [12] may be sample inconsistent, since they place the reconstructed signal into the guiding subspace. In contrast, [13] puts the reconstructed signal into the sampling subspace, relaxing the property that $\hat{\mathbf{f}} \in \mathcal{T}$ by minimizing instead the energy in $\mathcal{T}^\perp$. The reconstructed signal is defined as a point in the sample consistent plane $\mathbf{Sf} + \mathcal{S}^\perp$ having the smallest distance to $\mathcal{T}$. This approach is motivated by a realization that in practical applications, such as bandwidth expansion of narrowband audio signals, it may be difficult to explicitly find a frame of or even choose a trustworthy target reconstruction subspace $\mathcal{T}$. Thus, the subspace $\mathcal{T}$ is used as a guide, not as a true target, trusting the sampling more than the guiding. In [13], the subspace $\mathcal{T}$ is obtained by learning from a database of relevant signals.

Regularization-based methods, suggested in [1], [5], determine the reconstructed signal by solving an unconstrained problem minimizing a weighted sum, using a regularization parameter. The regularization parameter needs to be chosen *a priori*. The authors of [1] assume existence and uniqueness of the intersection of the sample-consistent reconstruction plane $\mathbf{Sf} + \mathcal{S}^\perp$ with the guiding reconstruction subspace $\mathcal{T}$ for any original signal $\mathbf{f}$, the same way as in [14], [15].

C. Main contributions

In the present work, we always assume that the guiding reconstruction subspace $\mathcal{T}$ is available in some form, e.g., implicitly via an action of the corresponding (possibly approximate) orthogonal projector $\mathbf{T}$. If, in addition, it is advantageous to utilize a specific norm such as in problems (2) and (3), we also assume that this knowledge is already incorporated into a definition of the norm $\|\cdot\|$ of the Hilbert space $\mathcal{H}$ that sets the stage for our reconstruction setup. We formulate a least squares approach that allows an implicit, frame-less, and approximate description of $\mathcal{T}^\perp$, e.g., in a form of a filter function, approximately suppressing $\mathcal{T}$ components of a signal. Additionally, the least squares approach allows and can benefit from oversampling, as in the generalized reconstruction of [11], making our reconstruction algorithms more stable, compared to classical constrained reconstructions using oblique projectors in [6], [7]. We describe a unified view of consistent, generalized and regularization based reconstruction methods. Conditions of existence and uniqueness of the reconstructed signal are obtained, using [16]. Moreover, stability and reconstruction error bounds are derived that improve on the bounds in [16]. The proofs of Theorems 4, 6, and 7 as well as numerical examples can be found in an extended journal version of this paper available from [4].

II. Overview of reconstruction in a Hilbert space

The intersection $\mathcal{S}^\perp \cap \mathcal{T}$ consists of signals in the guiding subspace $\mathcal{T}$ with zero samples, projections on $\mathcal{S}$. Its important role in the reconstruction is stated in the following assumption.

- **(A0)** *Reconstruction Uniqueness*: A reconstruction $\hat{\mathbf{f}}$ of a given signal $\mathbf{f}$ is unique if and only if $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$. Otherwise all possible reconstructions form the following closed plane $\hat{\mathbf{f}} + \{\mathcal{S}^\perp \cap \mathcal{T}\}$.

Possible basic assumptions on the reconstruction can be:

- **(A1)** *Sample Consistent*: The reconstructed signal yields the same sample as the original signal, i.e. $\mathbf{S}\hat{\mathbf{f}} = \mathbf{Sf},\ \forall \mathbf{f}$.
- **(A2)** *Sample Sufficient*: The reconstructed signal is fully determined, up to signals in $\mathcal{S}^\perp \cap \mathcal{T}$, by the sample of the original signal, i.e. $\hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_2 \in \mathcal{S}^\perp \cap \mathcal{T},\ \forall \mathbf{f}_1$ and $\mathbf{f}_2$ such that $\mathbf{Sf}_1 = \mathbf{Sf}_2$.
- **(A3)** *Guiding Subspace Reconstruction*: Signals in the guiding reconstruction subspace are reconstructed within the subspace, i.e. $\hat{\mathbf{f}} \in \mathcal{T},\ \forall \mathbf{f} \in \mathcal{T}$.
- **(A4)** *Reconstruction Stability*: A small change in the original signal results in a proportionally small change in the reconstructed signal, up to signals in $\mathcal{S}^\perp \cap \mathcal{T}$.

Axioms **(A1)** and **(A2)** imply that repeated reconstruction does not change, up to signals in $\mathcal{S}^\perp \cap \mathcal{T}$, an already reconstructed signal, i.e. $\hat{\mathbf{f}}_2 - \mathbf{f}_2 \in \mathcal{S}^\perp \cap \mathcal{T},\ \forall \mathbf{f}_2$ such that $\mathbf{f}_2 = \hat{\mathbf{f}}_1$, for an arbitrary $\mathbf{f}_1$. Indeed, $\mathbf{S}\hat{\mathbf{f}}_1 = \mathbf{Sf}_1$ by **(A1)**, so let us denote $\mathbf{f}_3 = \mathbf{S}\hat{\mathbf{f}}_1 = \mathbf{Sf}_1$. Axiom **(A2)** gives $\hat{\mathbf{f}}_2 - \hat{\mathbf{f}}_3 \in \mathcal{S}^\perp \cap \mathcal{T}$, using $\mathbf{f}_3 = \mathbf{S}\hat{\mathbf{f}}_1 = \mathbf{Sf}_2$, and $\hat{\mathbf{f}}_3 - \hat{\mathbf{f}}_1 \in \mathcal{S}^\perp \cap \mathcal{T}$, using $\mathbf{f}_3 = \mathbf{Sf}_1$, thus $\hat{\mathbf{f}}_2 - \hat{\mathbf{f}}_1 \in \mathcal{S}^\perp \cap \mathcal{T}$, which proves the claim.

Axioms **(A1)** and **(A3)** imply full conditional reconstruction, where signals in the guiding reconstruction subspace are exactly reconstructed, up to signals in $\mathcal{S}^\perp \cap \mathcal{T}$, i.e. we have that $\hat{\mathbf{f}} - \mathbf{f} \in \mathcal{S}^\perp \cap \mathcal{T},\ \forall \mathbf{f} \in \mathcal{T}$. Indeed, **(A1)** is equivalent to $\hat{\mathbf{f}} - \mathbf{f} \in \mathcal{S}^\perp,\ \forall \mathbf{f} \in \mathcal{H}$; at the same time, **(A3)** is equivalent to $\hat{\mathbf{f}} - \mathbf{f} \in \mathcal{T},\ \forall \mathbf{f} \in \mathcal{T}$. Thus, $\hat{\mathbf{f}} - \mathbf{f} \in \mathcal{S}^\perp \cap \mathcal{T},\ \forall \mathbf{f} \in \mathcal{T}$.

On the one hand, we want to define a reconstruction operator $\mathbf{R}: \mathcal{H} \to \mathcal{H}$, i.e. the reconstructed signal $\hat{\mathbf{f}}$ of $\mathbf{f}$ is given by $\hat{\mathbf{f}} = \mathbf{Rf}$, which requires uniqueness of $\hat{\mathbf{f}}$. On the other hand, the nontrivial intersection $\mathcal{S}^\perp \cap \mathcal{T} \neq \{\mathbf{0}\}$ naturally appear in some applications; see, e.g., [14]. Not having additional information, one cannot decide if any one reconstruction from the plane $\hat{\mathbf{f}} + \{\mathcal{S}^\perp \cap \mathcal{T}\}$ is better or worse than another, according to **(A0)**. Mathematically, we can resolve the issue by replacing the space $\mathcal{H}$ with a quotient-space $\mathcal{H}/\{\mathcal{S}^\perp \cap \mathcal{T}\}$, collapsing $\mathcal{S}^\perp \cap \mathcal{T}$ into zero, and consistently replacing the subspaces $\mathcal{S}$ and $\mathcal{T}$ with similar quotient-spaces. After such replacements, we have $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$, which we assume in the rest of this section, so the reconstruction operator $\mathbf{R}$ is defined by $\hat{\mathbf{f}} = \mathbf{Rf}$.

Sometimes, no target or even guiding reconstruction subspace is available or known at all, so $\mathbf{Rf} \in \mathcal{T}$ is inapplicable and replaced with signal energy minimization. E.g., in [8] the reconstructed signal $\hat{\mathbf{f}} = \mathbf{Rf}$ is determined as a solution of the following constrained minimization problem

$$\inf_{\hat{\mathbf{f}}} \|\mathbf{H}\hat{\mathbf{f}}\| \text{ subject to } \mathbf{S}\hat{\mathbf{f}} = \mathbf{Sf}, \tag{2}$$

where $\|\mathbf{H}\hat{\mathbf{f}}\|$ is an $\mathbf{H}$-dependent norm of the signal $\hat{\mathbf{f}}$ for a nonsingular operator $\mathbf{H}$. According to [8], problem (2) is equivalent to minimizing the worst case reconstruction error, i.e.

$$\inf_{\hat{\mathbf{f}}} \sup_{\mathbf{y} \in \mathcal{T}} \|\hat{\mathbf{f}} - \mathbf{y}\|,$$
$$\text{where } \mathcal{T} = \{\mathbf{y} : \mathbf{Sy} = \mathbf{Sf} \text{ and } \|\mathbf{Hy}\| \leq U\}, \tag{3}$$

and the solution does not depend on the constant $U > 0$.

III. Proposed Reconstruction Methods

We propose a novel formulation and algorithms for the sample consistent reconstruction, used in [13], which relaxes the constraint that $\hat{\mathbf{f}} \in \mathcal{T}$, used in [6], [7], instead minimizing the energy in $\mathcal{T}^\perp$, consistently with the sample, as in **(A1)**. Specifically, the reconstructed signal $\hat{\mathbf{f}}$ is determined as a solution of the following constrained minimization problem

$$\inf_{\hat{\mathbf{f}}} \|\hat{\mathbf{f}} - \mathbf{T}\hat{\mathbf{f}}\| \text{ subject to } \mathbf{S}\hat{\mathbf{f}} = \mathbf{Sf}, \tag{4}$$

which is equivalent to the problem

$$\inf_{\hat{\mathbf{x}} \in \mathcal{S}^\perp} \left\langle (\hat{\mathbf{x}} + \mathbf{Sf}), \mathbf{T}^\perp (\hat{\mathbf{x}} + \mathbf{Sf}) \right\rangle, \tag{5}$$

where $\hat{\mathbf{x}} = \hat{\mathbf{f}} - \mathbf{Sf}$. If the solutions $\hat{\mathbf{f}}$ and $\hat{\mathbf{x}}$ to problems (4) and (5), correspondingly, are not unique, we choose solutions in the corresponding factor-spaces, e.g., the normal (i.e. with the smallest norm) solutions $\hat{\mathbf{f}}_n$ and $\hat{\mathbf{x}}_n$ to guarantee the uniqueness required to define the reconstruction operator $\mathbf{R}$.

Equivalently, problem (5) has the following operator form,

$$\left(\mathbf{S}^\perp \mathbf{T}^\perp\right)\big|_{\mathcal{S}^\perp} \mathbf{x} = -\mathbf{S}^\perp \mathbf{T}^\perp \mathbf{Sf}, \tag{6}$$

where $(\cdot)\,|_{\mathcal{S}^\perp}$ denotes the operator restriction to its invariant subspace $\mathcal{S}^\perp$ (i.e. the domain of $\mathbf{S}^\perp\mathbf{T}^\perp$ is restricted to $\mathcal{S}^\perp$). If $\hat{\mathbf{x}}$ is a solution to the above problem, then the reconstructed signal $\hat{\mathbf{f}} = \hat{\mathbf{x}} + \mathbf{Sf}$ satisfies

$$\mathbf{S}^\perp \mathbf{T}^\perp \hat{\mathbf{f}} = \mathbf{0} \text{ and } \mathbf{S}\hat{\mathbf{f}} = \mathbf{Sf}, \tag{7}$$

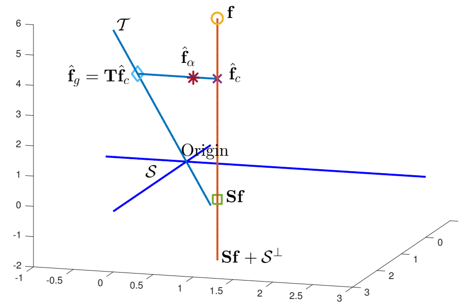

Fig. 1: 3D example showcasing the reconstruction interval between the sample consistent reconstruction $\hat{\mathbf{f}}_c$ and the generalized reconstruction $\hat{\mathbf{f}}_g = \mathbf{T}\hat{\mathbf{f}}_c$. The proposed reconstruction $\hat{\mathbf{f}}_\alpha$ exists anywhere on the reconstruction interval.

which is an operator form of our constrained minimization (4).

System of equations (7) is a particular case of the following system, investigated in [16], see also [9],

$$\mathbf{S}^\perp(\mathbf{A}\hat{\mathbf{f}} - \mathbf{g}) = \mathbf{0} \text{ and } \mathbf{S}(\hat{\mathbf{f}} - \mathbf{f}) = \mathbf{0}, \tag{8}$$

where $\mathbf{A}$ is a bounded self-adjoint non-negative operator on $\mathcal{H}$, i.e. $\mathbf{A} = \mathbf{A}^\star \geq 0$. When $\mathbf{g} = \mathbf{0}$ and $\mathbf{A} = \mathbf{T}^\perp$, we get system (7) and $N(\mathbf{A}) = \mathcal{T}$. If we split $\hat{\mathbf{f}}$ as in (1) then system (8) is equivalent to

$$\left(\mathbf{S}^\perp \mathbf{A}\right)\big|_{\mathcal{S}^\perp} \mathbf{x} = \mathbf{S}^\perp\left(\mathbf{g} - \mathbf{ASf}\right). \tag{9}$$

Conditions for existence and uniqueness of the solutions of equations (8) and (9) derived in [16] can be adapted for our reconstruction problem.

If the sample consistent reconstruction $\hat{\mathbf{f}}$ exists and is unique, we define a reconstruction set as a closed interval with the end points $\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^\perp$ and $\mathbf{t} = \mathbf{T}\hat{\mathbf{f}} \in \mathcal{T}$. If we trust that the sample-consistent closed plane $\mathbf{Sf} + \mathcal{S}^\perp$ is actually accurate, we choose our reconstruction to be sample consistent, $\hat{\mathbf{f}} \in \mathbf{Sf} + \mathcal{S}^\perp$, given by the first end point that solves, e.g., minimization problem (4). If there is noise in sample measurements, we may decide to trust the guiding closed subspace $\mathcal{T}$ more than the sample $\mathbf{Sf}$ and choose as our output reconstruction a convex linear combination $\hat{\mathbf{f}}_\alpha := \alpha\hat{\mathbf{f}} + (1-\alpha)\mathbf{T}\hat{\mathbf{f}}$ within the reconstruction set, where $0 \leq \alpha < 1$. The other extreme choice $\alpha = 0$ gives the strictly guided reconstruction $\mathbf{T}\hat{\mathbf{f}}$ suggested in [11]. Fig. 1 shows a 3D illustration of the proposed reconstruction $\hat{\mathbf{f}}_\alpha$ and its relation to the sample consistent reconstruction $\hat{\mathbf{f}}_c := \hat{\mathbf{f}}$ and the generalized reconstruction $\hat{\mathbf{f}}_g = \mathbf{T}\hat{\mathbf{f}}_c$.

The reconstruction based on solving (4) satisfies assumptions **(A0)**, **(A1)**, and **(A2)** by design. In the next section, we provide mathematical background, taking advantage of a theory developed in [16], that is then used to address the issues of existence, uniqueness, and to prove **(A3)** and **(A4)**.

IV. Uniqueness of the Reconstructed Signal

The following theorem gives a uniqueness condition of our reconstruction $\hat{\mathbf{f}}$.

Theorem 1. *(Based on [16, Lemma 4.2]) Let* $\hat{\mathbf{x}} \in \mathcal{S}^\perp$ *be a solution of* (6) *and* $\hat{\mathbf{f}} = \hat{\mathbf{x}} + \mathbf{Sf}$ *be a solution of* (7). *The solutions* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{f}}$ *are unique if and only if* $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$. *Otherwise, all solutions form a plane* $\hat{\mathbf{x}} + \{\mathcal{S}^\perp \cap \mathcal{T}\}$ *for* (6) *and a plane* $\hat{\mathbf{f}} + \{\mathcal{S}^\perp \cap \mathcal{T}\}$ *for* (7). *There exists unique normal solutions (with minimal norm in* $\mathcal{H}$*)* $\hat{\mathbf{x}}_n \in \mathcal{S}^\perp$ *of* (6) *and* $\hat{\mathbf{f}}_n$ *of* (7)*, which belong to the intersection of the corresponding plane and the closed subspace* $\left(\mathcal{S}^\perp \cap \mathcal{T}\right)^\perp = \overline{\mathcal{S}^\perp + \mathcal{T}}$*, and where* $\hat{\mathbf{f}}_n = \hat{\mathbf{x}}_n + \mathbf{Sf}$.

Theorem 2. *Reconstruction method* (5) *satisfies* ***(A3)****.*

Proof. According to **(A3)**, $\mathbf{f} \in \mathcal{T}$, but then $\hat{\mathbf{f}} = \hat{\mathbf{x}} + \mathbf{Sf} = \mathbf{f} \in \mathcal{T}$ is a solution of (4), since the minimizing quantity turns into zero, which is its smallest possible value. By Theorem 1, all solutions of (7) form the plane $\hat{\mathbf{f}} + \{\mathcal{S}^\perp \cap \mathcal{T}\} \subseteq \mathcal{T}$, since $\hat{\mathbf{f}} = \mathbf{f} \in \mathcal{T}$ and $\{\mathcal{S}^\perp \cap \mathcal{T}\} \subseteq \mathcal{T}$. ■

By Theorem 1, if $\mathcal{S}^\perp \cap \mathcal{T} \neq \{\mathbf{0}\}$, the solution $\hat{\mathbf{x}}$ to the reconstruction problem (6) and the reconstruction $\hat{\mathbf{f}} = \hat{\mathbf{x}} + \mathbf{Sf}$ determined by (7) are both not unique, and vice versa, consistently with the assumption **(A0)**. This can happen, e.g., if the number of samples is too small or when the guiding reconstruction subspace is too large. A similar issue appears in [14], dealing with non-unique strictly consistent reconstructions in $\mathcal{T}$ by choosing a subspace in $\mathcal{T}$, i.e. constraining the guiding reconstruction space. Instead, we propose constraining the orthogonal complement of the sampling subspace $\mathcal{S}$.

V. Existence and Stability

We begin by stating conditions for wellposedness, i.e. existence and stability of a solution, of problem (8) since it is later used to give us a bound on a reconstruction error. We denote operator $\mathbf{K} = \left(\mathbf{S}^\perp \mathbf{A}\right)\big|_{\mathcal{S}^\perp}$ obtaining

$$R(\mathbf{K}) = \mathbf{S}^\perp \mathbf{A} \mathcal{S}^\perp,\ N(\mathbf{K}) = N(\mathbf{S}^\perp \mathbf{A}) \cap \mathcal{S}^\perp = N(\mathbf{A}) \cap \mathcal{S}^\perp.$$

A normal solution of equation $\mathbf{Kx} = \mathbf{b}$ depends continuously on $\mathbf{b} \in R(\mathbf{K})$ if and only if the pseudo-inverse operator $\mathbf{K}^\dagger : R(\mathbf{K}) \to \mathcal{S}^\perp / N(\mathbf{K})$ is bounded. Here $\mathcal{S}^\perp / N(\mathbf{K})$ denotes the quotient space such that $\mathbf{y}, \mathbf{z} \in \mathcal{S}^\perp$ are equivalent if and only if $\mathbf{y} - \mathbf{z} \in N(\mathbf{K})$. The operator $\mathbf{K}^\dagger$ is bounded iff $R(\mathbf{K})$ is closed. The following theorem restates these conditions in terms of $\mathbf{A}$ and $\mathbf{S}$ for problem (8).

Theorem 3. *(Based on [16, Theorem 4.3]) A normal solution* $\hat{\mathbf{f}}_n = \hat{\mathbf{x}}_n + \mathbf{Sf}$ *to* (8) *with* $\hat{\mathbf{x}}_n = \mathbf{S}^\perp \hat{\mathbf{f}} \in \mathbf{S}^\perp R(\mathbf{A})$ *exists and depends continuously on arbitrary* $\mathbf{g} \in R(\mathbf{A}) + \mathcal{S}$ *and* $\mathbf{f} \in \mathcal{H}$ *if and only if*

$$\frac{1}{\rho} := \inf_{\mathbf{x} \in \mathbf{S}^\perp R(\mathbf{A}),\ \mathbf{x} \neq \mathbf{0}} \frac{\langle \mathbf{x}, \mathbf{Ax} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} > 0 \tag{10}$$

Moreover, condition (10) *implies*

$$\|\hat{\mathbf{x}}_n\|^2 \leq \rho^2 \|\mathbf{S}^\perp(\mathbf{g} - \mathbf{ASf})\|^2, \tag{11}$$

that also leads to an upper bound for $\|\hat{\mathbf{f}}_n\|^2 = \|\mathbf{Sf}\|^2 + \|\hat{\mathbf{x}}_n\|^2$.

Taking $\mathbf{g} = \mathbf{0}$ and $\mathbf{A} = \mathbf{T}^\perp$, we obtain system (7) and $N(\mathbf{A}) = \mathcal{T}$. Condition (10) with $\mathbf{A} = \mathbf{T}^\perp$ is equivalent to

$$\kappa := \inf_{\mathbf{x} \in \mathbf{S}^\perp \mathcal{T}^\perp} \frac{\|\mathbf{T}^\perp \mathbf{x}\|}{\|\mathbf{x}\|} > 0, \text{ where } \rho = \frac{1}{\kappa^2}, \tag{12}$$

which becomes the key assumption. Let us describe (12) via concepts of the minimal gap γ and angles Θ between subspaces.

Theorem 4. *(Based on [16, Lemma 4.6]) Let* κ *be defined by* (12). *Then*

$$\kappa = \gamma(\mathcal{S}, \mathcal{T}^\perp) = \cos\left(\theta_{\max}\right),$$

where

$$\gamma\left(\mathcal{S}, \mathcal{T}^\perp\right) := \inf_{\mathbf{f} \in \mathcal{S}, \mathbf{f} \notin \mathcal{T}^\perp} \frac{dist\left(\mathbf{f}, \mathcal{T}^\perp\right)}{dist\left(\mathbf{f}, \mathcal{S} \cap \mathcal{T}^\perp\right)}, \tag{13}$$

is the minimal gap between closed subspaces $\mathcal{S}$ *and* $\mathcal{T}^\perp$*, and*

$$\theta_{\max} = \sup\{\Theta(\mathcal{S}, \mathcal{T}) \setminus \{\pi/2\}\}, \tag{14}$$

is the largest non-trivial angle between closed subspaces $\mathcal{S}$ *and* $\mathcal{T}$.

Let us also note that by [16, Lemma 4.6] we have

$$\gamma(\mathcal{S}, \mathcal{T}^\perp) = \gamma(\mathcal{T}, \mathcal{S}^\perp) = \inf_{\mathbf{x} \in \mathbf{T}\mathcal{S}} \frac{\|\mathbf{Sx}\|}{\|\mathbf{x}\|}. \tag{15}$$

The assumption $\gamma(\mathcal{T}, \mathcal{S}^\perp) > 0$ is equivalent to assuming that the sum $\mathcal{S}^\perp + \mathcal{T}$ is closed. The latter is automatically satisfied if $\mathcal{S}^\perp + \mathcal{T} = \mathcal{H}$ as traditionally assumed in reconstruction literature; see, e.g., [6], [7]. Theorems 3 and 4 imply

Theorem 5. *If* $\cos\theta_{\max} > 0$*, then there exists a solution of the reconstruction problem* (7) *for any signal* $\mathbf{f}$*; the normal solution* $\hat{\mathbf{f}}_n$ *of* (7) *is unique and bounded by*

$$\|\hat{\mathbf{f}}_n\|^2 \leq \|\mathbf{Sf}\|^2 + \|\mathbf{S}^\perp \mathbf{T}^\perp \mathbf{Sf}\|^2 / \cos^4\theta_{\max}. \tag{16}$$

Let us note that Theorem 5 applies Theorem 3 with $g = 0$ and leaves open a question whether condition (10) or condition (12) is still necessary in this case. In the rest of the section, we go beyond the results presented in [16] and address this question, using a powerful theory for a pair of two orthogonal projectors; see, e.g., [17].

Theorem 6. *We denote by* $\mathcal{H}_0$ *the subspace of* $\mathcal{H}$ *that is orthogonal to all four subspaces* $\mathcal{S} \cap \mathcal{T}$, $\mathcal{S}^\perp \cap \mathcal{T}$, $\mathcal{S} \cap \mathcal{T}^\perp$*, and* $\mathcal{S}^\perp \cap \mathcal{T}^\perp$*, as introduced in [18]. Let* $\mathbf{P}_0$ *be the orthogonal projector onto the subspace* $\mathcal{H}_0$. *The assumption* $\cos\theta_{\max} > 0$ *is necessary and sufficient for existence of a solution of the reconstruction problem* (7) *for any signal* $\mathbf{f}$. *A normal solution* $\hat{\mathbf{x}}_n$ *to* (6)*, giving the normal reconstruction* $\hat{\mathbf{f}}_n = \hat{\mathbf{x}}_n + \mathbf{Sf}$ *exists and depends continuously on arbitrary* $\mathbf{f} \in \mathcal{H}$ *if and only if* $\cos\theta_{\max} > 0$. *If* $\cos\theta_{\max} > 0$*, bound* (16) *holds, as well as*

$$\|\hat{\mathbf{x}}_n\| \leq \|\mathbf{T}^\perp \mathbf{SP}_0 \mathbf{f}\| / \cos\theta_{\max} \tag{17}$$

and

$$\|\hat{\mathbf{x}}_n\| \leq \|\mathbf{SP}_0 \mathbf{f}\| \tan\theta_{\max} \tag{18}$$

in $\|\hat{\mathbf{f}}_n\|^2 = \|\mathbf{Sf}\|^2 + \|\hat{\mathbf{x}}_n\|^2$.

We finally note that neither of the bounds (16), (17), and (18) can be derived from the other one, i.e. none of them is in general sharper than the other.

VI. Reconstruction Error Bounds

If the original signal satisfies $\mathbf{f} \in \mathcal{T}$ and $\mathcal{S}^\perp \cap \mathcal{T} = \{\mathbf{0}\}$, then the proposed reconstruction (7) perfectly recovers it. Suppose now that we obtain a reconstruction $\hat{\mathbf{f}}$ of some $\mathbf{f} \notin \mathcal{T}$ by solving (7). An important question in this context is to bound the reconstruction error $\hat{\mathbf{f}} - \mathbf{f}$.

If $\mathcal{S}^\perp \cap \mathcal{T} \neq \{\mathbf{0}\}$, then the solution to reconstruction problem (7) is evidently not unique. In this case, it is still possible to bound the reconstruction error, but in the factor space $\mathcal{H}/\left(\mathcal{S}^\perp \cap \mathcal{T}\right)$. Let $\mathbf{M}$ be an orthogonal projector onto $\left(\mathcal{S}^\perp \cap \mathcal{T}\right)^\perp = \overline{\mathcal{S} + \mathcal{T}^\perp}$, such that $\mathbf{M} = \mathbf{P}_\mathcal{H} - \mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}$. Then the norm of the error in the factor space equals the norm of its projection on the subspace $\overline{\mathcal{S} + \mathcal{T}^\perp}$, representing the factor space $\mathcal{H}/\left(\mathcal{S}^\perp \cap \mathcal{T}\right)$. In other words, we need to bound above the quantity $\left\|M\left(\hat{\mathbf{f}} - \mathbf{f}\right)\right\|$, removing from the consideration the $\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}\mathbf{f}$ part of the original signal $\mathbf{f}$ and ignoring the non-unique part $\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}\hat{\mathbf{f}}$ of the reconstructed signal $\hat{\mathbf{f}}$. If the uniqueness condition holds, we have $\left(\mathcal{S}^\perp \cap \mathcal{T}\right)^\perp = \mathcal{H}$ and $\mathbf{M}\left(\hat{\mathbf{f}} - \mathbf{f}\right) = \hat{\mathbf{f}} - \mathbf{f}$.

The unique normal solution $\hat{\mathbf{f}}_n$ of problem (7) simply drops the $\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}\mathbf{f}$ part of the original signal $\mathbf{f}$, Thus, the term $\|\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}\mathbf{f}\|$ appears in the upper bound for $\|\hat{\mathbf{f}}_n - \mathbf{f}\|$, but not for $\left\|M\left(\hat{\mathbf{f}} - \mathbf{f}\right)\right\|$.

The $\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}^\perp}\mathbf{f}$ part of the original signal $\mathbf{f}$ is visible neither in the sample $\mathbf{Sf}$, nor to the guiding orthoprojector $\mathbf{T}$, thus the term $\|\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}^\perp}\mathbf{f}\|$ is expected in any error bound.

The following theorem gives reconstruction error bounds.

Theorem 7. *Let* $\cos\theta_{\max} > 0$. *In the notation of Theorem 6, let us consider the normal solution* $\hat{\mathbf{x}}_n$ *to* (6), *giving the normal reconstruction* $\hat{\mathbf{f}}_n = \hat{\mathbf{x}}_n + \mathbf{Sf}$ *as well as any reconstruction* $\hat{\mathbf{f}}$, *obtained by solving* (7). *Let* $\mathbf{M}$ *be the orthoprojector onto* $\overline{\mathcal{S} + \mathcal{T}^\perp}$ *and* $\mathbf{P}_0$ *be defined as in Theorem 6. Then,*

$$\left\|M\left(\hat{\mathbf{f}} - \mathbf{f}\right)\right\|^2 = \|\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}^\perp}\mathbf{f}\|^2 + \left\|\hat{\mathbf{x}}_n - \mathbf{S}^\perp\mathbf{P}_0\mathbf{f}\right\|^2$$

and

$$\left\|\hat{\mathbf{f}}_n - \mathbf{f}\right\|^2 = \|\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}}\mathbf{f}\|^2 + \|\mathbf{P}_{\mathcal{S}^\perp \cap \mathcal{T}^\perp}\mathbf{f}\|^2 + \left\|\hat{\mathbf{x}}_n - \mathbf{S}^\perp\mathbf{P}_0\mathbf{f}\right\|^2,$$

and the following bounds hold

$$\left\|\hat{\mathbf{x}}_n - \mathbf{S}^\perp\mathbf{P}_0\mathbf{f}\right\| \leq \|\mathbf{S}^\perp\mathbf{T}^\perp\mathbf{P}_0\mathbf{f}\|/\cos^2\theta_{\max}, \qquad (19)$$

and

$$\left\|\hat{\mathbf{x}}_n - \mathbf{S}^\perp\mathbf{P}_0\mathbf{f}\right\| \leq \|\mathbf{T}^\perp\mathbf{P}_0\mathbf{f}\|/\cos\theta_{\max}. \qquad (20)$$

The error bounds of Theorem 7 based on (20), extend to the most general case the one (which is $\|\mathbf{T}^\perp\mathbf{f}\|/\cos\theta_{\max}$) obtained with the consistent reconstruction method presented in [6], [12], dropping all unnecessary assumptions on the sampling and guiding subspaces made in [6], [12]. The error bounds of Theorem 7 based on (19) are new. We finally note that neither of the bounds (19) and (20) can be derived from the other one.

VII. Conclusion

Our efficient reconstruction algorithms allow reconstructing signals with desired properties given by a guiding subspace. The proposed methodology is very general and is expected to be effective for a wide range of applications, in video and audio processing, data mining, and real time security and artificial intelligence systems.

References

[1] S. K. Narang, A. Gadde, E. Sanou, and A. Ortega, “Localized iterative methods for interpolation in graph structured data,” in *2013 IEEE Global Conference on Signal and Information Processing*, Dec 2013, pp. 491–494, doi:10.1109/GlobalSIP.2013.6736922.

[2] X. Wang, J. Chen, and Y. Gu, “Generalized graph signal sampling and reconstruction,” in *2015 IEEE Global Conference on Signal and Information Processing, GlobalSIP 2015, Orlando, FL, USA, December 14-16, 2015*, 2015, pp. 567–571, doi:10.1109/GlobalSIP.2015.7418259.

[3] A. Gadde, A. Anis, and A. Ortega, “Active semi-supervised learning using sampling theory for graph signals,” in *Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, ser. KDD ’14. New York, NY, USA: ACM, 2014, pp. 492–501, doi:10.1145/2623330.2623760.

[4] A. Knyazev, A. Gadde, H. Mansour, and D. Tian, “Guided signal reconstruction theory,” Tech. Rep., February. [Online]. Available: http://arxiv.org/abs/1702.00852

[5] A. Kheradmand and P. Milanfar, “A general framework for regularized, similarity-based image restoration,” *IEEE Transactions on Image Processing*, vol. 23, no. 12, pp. 5136 – 5151, 2014, doi:10.1109/TIP.2014.2362059.

[6] M. Unser and A. Aldroubi, “A general sampling theory for nonideal acquisition devices,” *Signal Processing, IEEE Transactions on*, vol. 42, no. 11, pp. 2915–2925, 1994, doi:10.1109/78.330352.

[7] Y. C. Eldar, “Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors,” *Journal of Fourier Analysis and Applications*, vol. 9, no. 1, pp. 77–96, 2003, doi:10.1007/s00041-003-0004-2.

[8] Y. C. Eldar and T. Michaeli, “Beyond bandlimited sampling,” *Signal Processing Magazine, IEEE*, vol. 26, no. 3, pp. 48–68, 2009, doi:10.1109/MSP.2009.932125.

[9] M. L. Arias, G. Corach, and M. C. Gonzalez, “Saddle point problems, Bott-Duffin inverses, abstract splines and oblique projections,” *Linear Algebra and Its Applications*, vol. 457, no. Complete, pp. 61–75, 2014, doi:10.1016/j.laa.2014.05.006.

[10] T. Kato, *Perturbation theory for linear operators*. Berlin: Springer, 1995, doi:10.1007/978-3-642-66282-9.

[11] B. Adcock and A. C. Hansen, “A generalized sampling theorem for stable reconstructions in arbitrary bases,” *Journal of Fourier Analysis and Applications*, vol. 18, no. 4, pp. 685–716, 2012, doi:10.1007/s00041-012-9221-x.

[12] Y. C. Eldar and T. G. Dvorkind, “A minimum squared-error framework for generalized sampling,” *Signal Processing, IEEE Transactions on*, vol. 54, no. 6, pp. 2155–2167, 2006, doi:10.1109/TSP.2006.873488.

[13] D. Bansal, B. Raj, and P. Smaragdis, “Bandwidth expansion of narrowband speech using non-negative matrix factorization,” in *Ninth European Conference on Speech Communication and Technology*, 2005. [Online]. Available: http://www.merl.com/reports/docs/TR2005-135.pdf

[14] A. Hirabayashi and M. Unser, “Consistent sampling and signal recovery,” *Signal Processing, IEEE Transactions on*, vol. 55, no. 8, pp. 4104–4115, Aug 2007, doi:10.1109/TSP.2007.895996.

[15] G. Corach and J. Giribet, “Oblique projections and sampling problems,” *Integral Equations and Operator Theory*, vol. 70, no. 3, pp. 307–322, 2011, doi:10.1007/s00020-010-1858-8.

[16] A. V. Knyazev, “Observations on degenerate saddle point problems,” *Computer methods in applied mechanics and engineering*, vol. 196, no. 37, pp. 3742–3749, 2007, doi:10.1016/j.cma.2006.10.019.

[17] A. Knyazev, A. Jujunashvili, and M. Argentati, “Angles between infinite dimensional subspaces with applications to the Rayleigh–Ritz and alternating projectors methods,” *Journal of Functional Analysis*, vol. 259, no. 6, pp. 1323–1345, 2010, doi:10.1016/j.jfa.2010.05.018.

[18] P. R. Halmos, “Two subspaces,” *Trans. Amer. Math. Soc.*, vol. 144, pp. 381–389, 1969, doi:10.1090/S0002-9947-1969-0251519-5.